Mechanical Energy : KE + APE

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Energy can be a useful way to understand fluid systems because all terms are expressed in the same units : Joules and Watts.

- Kinetic energy ≡ work done to accelerate a mass m to speed $|\underline{u}|$.
- Rate of doing work = force × velocity (*)
- In fluid systems we use $\frac{\text{mass}}{\text{unit volume}} \cdot \text{acceleration} = \frac{\text{force}}{\text{unit volume}}$

or $\rho \frac{D\underline{u}}{Dt} = \text{Force}/\text{vol.}$

So to evaluate (*) following a fluid parcel as it accelerates from 0 to speed $|\underline{u}|$:

$$\int_{t_0}^{t_1} \frac{\text{force}}{\text{vol.}} \cdot \underline{u}\, dt = \rho \int_{t_0}^{t_1} \frac{D\underline{u}}{Dt} \cdot \underline{u}\, dt = \rho \int_{t_0}^{t_1} \frac{D}{Dt}\left(\tfrac{1}{2}\underline{u}^2\right) dt$$

$= \frac{1}{2}\rho \underline{u}^2$ assuming $\underline{u} = 0$ at $t = t_0$

this is $\frac{KE}{\text{unit vol.}}$ or "KE_v", units $\frac{J}{m^3} = \frac{kg\, m^2}{s^2}\frac{1}{m^3} = \frac{kg}{m s^2}$

and often we casually (imprecisely) say $KE = \frac{1}{2}\underline{u}^2 \sim \frac{m^2}{s^2}$

- Note: our spatial integral for steady flow in the Bernoulli derivation gives the same result:

- Notation $\underline{x}^L$ = Lagrangian position of a fluid parcel

so each $\underline{x}^L$ corresponds to a specific time

$\Rightarrow$ e.g. at $t = t_0$ particle is at $\underline{x} = \underline{x}_0^L$

$$\rho \int_{t_0}^{t_1} \frac{D\underline{u}}{Dt} \cdot \underline{u} \, dt = \rho \int_{t_0}^{t_1} \frac{D\underline{u}}{Dt} \cdot \overbrace{\frac{d\underline{x}^L}{dt}}^{=\underline{u}} dt = \rho \int_{\underline{x}_0^L}^{\underline{x}_1^L} \frac{D\underline{u}}{Dt} \cdot d\underline{x}^L$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$= \rho \frac{1}{2} \underline{u}^2$ $\longleftarrow$ same $\longrightarrow$ $= \frac{1}{2} \rho \underline{u}^2$ (if $\frac{\partial}{\partial t} = 0$)

KE Derivation — Bernoulli derivation

Back to Energy:

For SW flow $KE_V = \frac{1}{2}\rho(u^2 + v^2)$ and

we commonly use $\frac{KE}{\text{unit horizontal area}}$ = "KE_A" $= \int_{-H}^{\eta} KE_V \, dz$

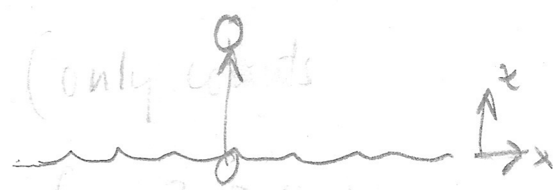
$\Rightarrow$ $\boxed{KE_A = \frac{1}{2}\rho h (u^2 + v^2) \quad \text{for SW flow}}$ Joules / m^2

• Available Potential Energy $\equiv$ work done against gravity (buoyancy) to move a mass from its rest state to its current location.

Note: be sure you can define what the APE is available to do.

To move the free surface away from its rest state at $z=0$, for a single water particle

$$APE_V = \int_0^z \rho g \, dz = \rho g z$$



only counts for above $z=0$.

To make the APE_A of a disturbed free surface we integrate over all such particles.

$$APE_A = \int_0^\eta \rho g \, z \, dz = \boxed{\tfrac{1}{2}\rho g \eta^2 = APE_A}$$

For downward displacement we are pushing air particles down through water - work done is still positive.

Forming governing equations for KE_A & APE_A

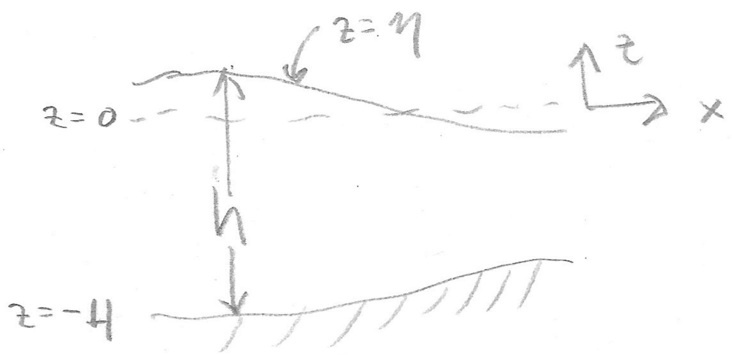
- unidirectional, frictional, SW flow
- don't linearize

(lead-in to hydraulics)

$$\rho h u \cdot \left[\ \boxed{x \text{ mom}} \quad u_t + u u_x + g \eta_x + \frac{C_d |u| u}{h} = 0 \ \right] \quad \text{(i)}$$

$$\rho g \eta \cdot \left[\ \boxed{\text{mass}} \quad \eta_t + (hu)_x = 0 \ \right] \quad \text{(ii)}$$

note $\eta \eta_t = (\frac{1}{2}\eta^2)_t$ e.g.



from (i): $\rho h (\frac{1}{2}u^2)_t + \rho h u (\frac{1}{2}u^2)_x + g \rho h u \eta_x + \rho C_d |u|^3$

or:

$$(\rho h \tfrac{1}{2}u^2)_t + (\rho h u \tfrac{1}{2}u^2)_x - (\tfrac{1}{2}\rho u^2)\underbrace{\cancel{\left[h_t + (hu)_x\right]}}_{\to 0 \text{ by } \boxed{\text{mass}}}$$

note $h_t = \eta_t$

$$+ (g \rho h u)\eta_x + \rho C_d |u|^3 = 0$$

these combine to form "pressure work"

from (ii)

$$(\tfrac{1}{2}\rho g \eta^2)_t + \rho g \eta (hu)_x = 0$$

Adding the results from (i) & (ii) we find the full SW energy equation

$$\underbrace{\left(\tfrac{1}{2}\rho h u^2 + \tfrac{1}{2}\rho g \eta^2\right)_t}_{\text{Rate of change of } KE_A \text{ \& } APE_A} + \left[u\left(\tfrac{1}{2}\rho h u^2\right) + \rho g \eta h u\right]_x = -\rho C_d |u|^3$$

(Divergence of) Advection of KE_A

(Divergence of) Pressure work ☆

Energy loss to bottom friction (neg. definite)

☆ Pressure work $= (\rho g \eta) \times (hu)$ = pressure anomaly × transport

- is how energy is transmitted over long distances by waves

- Except for very fast flow, Pressure Work is much larger than KE Advection:

Scaling $\dfrac{\text{KE Adv.}}{\text{Press. Work}} = \dfrac{1}{2}\dfrac{U^{\cancel{3}\,2} H}{a\,\underbrace{gH}_{c^2}\,\cancel{U}} \sim \dfrac{U^2}{c^2 \quad \underbrace{c\frac{a}{H}}_{\sim U \text{ from linear SW waves}}} \sim \dfrac{U}{c}$

This is called a "Froude Number" (velocity / wave speed)

For our linear SW waves;

Fr = Froude # $\sim \frac{u}{c} \sim \frac{a}{H} \ll 1$

$\Rightarrow$ KE advection unimportant

Also:

- $c_g = c_p = \sqrt{gH}$: non-dispersive
- Scales of KE_A + APE_A are equal ("equipartition")

  ☆ Prove this, using $u \sim c\frac{a}{H}$

- Can estimate the net energy flux through a channel of width B as

$$\text{Flux} \sim B\rho g H\langle \eta u \rangle$$

$\langle\ \rangle$ = tidal average

A • What are the units of the flux?

- What if η + u are out of phase?
- What does this tell you about net energy dissipation (say channel leads to a closed bay)?